SPIN OF AN ELECTRON FROM FIVE-DIMENSIONAL WAVE EQUATION

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ABSTRACT. The author attempts a general solution of five dimensional analogue of Klein-Gordon equation and finds that an intrinsic angular momentum of the electron similar to the spin appears as a consequence of the solution.

1. INTRODUCTION

In a previous paper the author (Banerjee, 1957) has obtained Sommerfeld's fine structure formula in exact form from five dimensional analogue of the Klein-Gordon wave equation. There it has been found that the introduction of the fifth co-ordinate influences the energy levels of the electron in the same way as the existence of its spin does it may be remarked here that the connection between the fifth co-ordinate of a particle and its spin is not, however, apparent. The idea to represent the motion of a particle in five-dimensional space-time continuum has engaged the attentions of Kaluza (1921), Klein (1926, 1946), Einstein (1931, 1932), Pauli (1933), Wilson (1928), Fisher (1929), Flint (1946), Corben (1952) and others , they have, however, confined themselves to purely theoretical lines It has been felt that it may be worthwhile to see how the idea fares with regard to its practical application and this feeling has enabled the author to obtain Sommerfeld's fine structure formula which result gives us the hope that this representation may throw more light on the mysteries of quantum phenomena It is further hoped that further investigation may give us better insight as to whether or how the fifth co-ordinate is connected to spin.

The author in the previous paper has obtained for the five dimensional equation a special solution when the two of the four quantum numbers coincided In the present paper that restriction has been removed and a general solution leading to the appearance of four quantum numbers has been obtained. It has been found that the square of the angular momentum has eigen values $j(j+1)\hbar^2$ where $j=l+\frac{1}{2}$; the appearance of half integers indicates that spin of the electron has entered into picture in a subtle way. The extra quantum number p, which appears here as a consequence of the additional co-ordinate, does not affect the energy levels of the electron subjected only to a coulomb field. It is necessary to pursue the line to find out what part the extra quantum number plays in atomic and nuclear processes.

2. GENERAL SOLUTION OF THE WAVE EQUATION

The relativistic wave equation in polar co-ordinates in five dimensions is given by

$$\frac{\partial^2\psi}{\partial r^2}+\frac{3}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2\sin^2\chi}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\chi\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$+\frac{1}{r^2\sin^2\chi}\frac{\partial}{\partial\chi}\left(\sin^2\chi\frac{\partial\psi}{\partial\chi}\right)+\left(A+\frac{2B}{r}+\frac{Z^2\alpha^2}{r^2}\right)\psi=0 \quad \ldots \quad (1)$$

where $A=\frac{W^2-m_0^2c^4}{\hbar^2c^2}$, $B=WZ\alpha^2$ and $\alpha=\frac{e^2}{\hbar c}$

(This equation is the same as the equation (4a) of the previous paper). We may separate the above equation into two equations given below

$$\frac{r^2}{\psi_r}\left[\frac{\partial^2\psi_r}{\partial r^2}+\frac{3}{r}\frac{\partial\psi_r}{\partial r}+\left(A+\frac{2B}{r}+\frac{Z^2\alpha^2}{r^2}\right)\psi_r\right]=l(l+2) \quad \ldots \quad (2)$$

and

$$\frac{1}{\psi_\theta}\frac{1}{\sin^2\chi\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_\theta}{\partial\theta}\right)+\frac{1}{\psi_\varphi}\frac{1}{\sin^2\chi\sin^2\theta}\frac{\partial^2\psi_\varphi}{\partial\phi^2}+\frac{1}{\psi_\chi}\frac{1}{\sin^2\chi}\frac{\partial}{\partial\chi}$$

$$\left(\sin^2\chi\frac{\partial\psi_\chi}{\partial\chi}\right)=-l(l+2) \quad \ldots \quad (3)$$

where l is a positive integer. The equation (2) leads to the fine structure formula in exact form which has been done in the previous paper and the equation (3) may be split up into two following equations.

$$\frac{1}{\psi_\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_\theta}{\partial\theta}\right)+\frac{1}{\psi_\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2\psi_\varphi}{\partial\phi^2}=-p(p+1) \quad \ldots \quad (4)$$

and

$$\frac{1}{\psi_\chi}\frac{\partial}{\partial\chi}\left(\sin^2\chi\frac{\partial\psi_\chi}{\partial\chi}\right)+l(l+2)\sin^2\chi=p(p+1) \quad \ldots \quad (5)$$

where p is a positive integer.
The equation (5) becomes on substitution $\psi_\chi=(\sin\chi)^{-\frac{1}{2}}\,\omega$ where ω is a function of χ,

$$(\sin\chi)^{-\frac{1}{2}}\left[\frac{1}{\sin\chi}\frac{\partial}{\partial\chi}\left(\sin\chi\frac{\partial\omega}{\partial\chi}\right)+(l+\tfrac{1}{2})(l+\tfrac{3}{2})\omega-\frac{(p+\frac{1}{2})^2}{\sin^2\chi}\omega\right]=0 \quad \ldots \quad (6)$$

Putting $j = l+\frac{1}{2}$ and $q = p+\frac{1}{2}$ and since $(\sin\chi)^{-1/2} \neq 0$ we have

$$\frac{1}{\sin\chi}\frac{\partial}{\partial\chi}\left(\sin\chi\frac{\partial\omega}{\partial\chi}\right)+j(j+1)\omega-\frac{q^2}{\sin^2\chi}\,\omega=0 \quad \ldots \quad (7)$$

The solution of the equation (7) is $P_j^q(\cos\chi)$ and this is expressible in the form of a rational integral function of $\cos\chi$ and $\sin\chi$ when $j-q$ is a positive integer (Thomson and Tait, 1879). Hence the time-independent ψ-function in five-dimensional continuum may be written as

$$\psi = \psi_r \cdot P_p^m(\cos\theta)\; e^{im\phi}\,(\sin\chi)^{-\frac{1}{2}}\; P_{l+1/2}^{p+1/2}(\cos\chi) \quad \ldots \quad (8)$$

where ψ_r denotes the solution of the radial equation (2). The above solution reduces to our solution in the previous paper when $l=p$. It is seen from the general solution given above that it contains an extra quantum number (p) which, however does not affect the eigen energies of the bound electron. Further the appearance of half integers makes one feel that the spin of the electron has been embedded in the formalism in a curious way.

3 INTRINSIC ANGULAR MOMENTUM OR SPIN

In three dimensional space (*i.e.* four dimensional continuum) the square of the angular momentum operator is given by

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \quad \ldots \quad (9a)$$

Replacing $\frac{\partial^2}{\partial\phi^2}$ by $-m^2$, we have

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{m^2}{\sin^2\theta}\right] \quad \ldots \quad (9b)$$

We maintain that in our four dimensional space (*i.e.* five dimensional continuum) the square of the angular momentum operator, which we shall denote by J^2 maintains the same form as in three dimensional space.

Now from equations (6) and (7) we have an operator

$$-\hbar^2\left[\frac{1}{\sin\chi}\frac{\partial}{\partial\chi}\left(\sin\chi\frac{\partial}{\partial\chi}\right)-\frac{q^2}{\sin^2\chi}\right]$$

which has the desired form and we associate this operator in our four dimensional space as the square of the angular momentum operator. The eigen values of this operator is evidently given by $j(j+1)\hbar^2$ and hence $J = \sqrt{j(j+1)}\hbar$. When $l = 0$, we have $J^2 = 3/4\,\hbar^2$, $J = \sqrt{\frac{1}{2}(\frac{1}{2}+1)}\hbar$ and $j\hbar = \hbar/2$. Since $l = 0$ corresponds to the state of the particle independent of θ, ϕ and χ we may assign these values to be due to the intrinsic angular momentum or what is called the spin of the particle

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